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RESEARCH MEMORANDUM

NOTES ON LINEAR PROGRAMMING: PART XXXVI:
THE ALLOCATION OF AIRCRAFT TO ROUTES --
AN EXAMPLE OF LINEAR PROGRAMMING UNDER
UNCERTAIN DEMAND

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SUMMARY

The purpose of this paper is to illustrate an application of linear programming to the problem of allocation of aircraft to routes in order to maximize expected profits when there is uncertain customer demand. The approach is intuitive; the theoretical basis of this work is found in an earlier study. The allocations are compared with those obtained under the usual procedure of assuming a fixed demand equal to the expected value. The computational procedure is similar to that of the fixed-demand case, with only slightly more computational effort required.

This paper is intended both for readers interested in routing problems (and analogous resource-allocation problems) and for those interested in studying an example of an application of linear programming under uncertainty.

CONTENTS

THE ALLOCATION OF AIRCRAFT TO ROUTES—AN EXAMPLE OF LINEAR PROGRAMMING UNDER UNCERTAIN DEMAND

1. INTRODUCTION

There are many business, economic, and military problems that have the following characteristics in common: a limited quantity of capital equipment or final product must be allocated among a number of final-use activities, where the level of demand for each of these activities, and hence the payoff, is uncertain; further, once the allocation is made, it is not economically feasible to reallocate because of geographical separation of the activities, because of differences in form of the final products, or because of a minimum lead time between the decision and its implementation. Examples of such problems are (1) the scheduling of transport vehicles over a number of routes to meet a demand in some future period and (2) the allocating of quantities of a commodity at discrete time intervals among several storage or distribution points while the future demand for the commodity is unknown. It is assumed, however, that demand can be forecast or estimated as a distribution of values, each with a specified probability of being the actual value.

The general area where the techniques of this paper apply may be schematized broadly as problems where:

1. Alternative sets of activity levels can be chosen consistent with given resources.
2. Each set of chosen activity levels provides the facilities or stocks to meet a demand that is itself unknown but that has a known frequency distribution.

3. Profits depend on the costs of the facilities, or stocks, and on the revenues from the demand.
4. The general objective is to determine that set of activity levels that maximizes profits.

The paper entitled "Linear Programming under Uncertainty" [2] forms the theoretical basis for the present work. Our purpose is to illustrate the procedural steps with the example that, in fact, originally prompted the referenced theoretical work in this area. Thus, little in the way of rigorous theory will be attempted in this paper, although each step will be justified intuitively.

The method is explained by the use of a model for routing aircraft. Several types of aircraft are allocated over a number of routes; the monthly demand for service over each route is assumed to be known only as a distribution of probable values. The aircraft are so allocated as to minimize the sum of (a) the cost of performing the transportation and (b) the expected value of the revenue lost through the failure to serve all the traffic that actually develops.

For purposes of month-to-month scheduling, an air-transport operator would, presumably, feel better about having to make an estimate of the range and general distribution of future travel (or shipments) over his routes than about having to commit himself to a single expected value. Indeed, he might feel that the optimal assignment should be insensitive to a wide range of demand distributions, and that an assignment based on expected values (as if these were known fixed demands) would be misl[illegible]

It is suggested that the reader make sensitivity tests by modifying the demand distributions given in the illustrative example.

Passenger demand, of course, occurs on a day-by-day—in fact, on a flight-by-flight basis. The assumed number of passengers per aircraft of a given type per flight on a given route may be thought of as an ideal number that can be increased slightly by decreasing the amount of air freight when this is indicated, and by "smoothing" the demand through encouraging the customers to take open reservations on alternative flights as opposed to less certain reservations on desired flights. In spite of these possible adjustments, traveler preferences and the inevitable last-minute cancellations do cause loss of passenger-carrying capability. However, the best way to reflect these effects of the daily variations in demand are beyond the scope of this paper. For our purpose here, either the aircraft passenger-carrying capability or the demand may be thought of as adjusted downward to reflect the loss due to daily variations of demand.

The method employed is simple, and the example used can be solved by hand in an hour or two. Larger problems can be solved with computing machines.

In a previously published paper [1], the method was applied to the same example, assuming the demand on each route to be known;* the present paper continues the analysis, showing how

*This was equivalent to using the expected value of demand, rather than taking account of the whole frequency distribution, as in the present paper.

to handle a frequency distribution of demand relative to each route. A different allocation is found to be optimal in this case.

We shall now describe the problem, briefly indicate the nature of the solution based on expected values, show the method of solving the problem using stochastic values for demand, and finally compare the two solutions.

2. REVIEW OF FIXED-DEMAND EXAMPLE

The fixed-demand example that we are using to illustrate the method takes a fixed fleet of four types of aircraft, as shown in Table 1.

Table 1

ASSUMED AIRCRAFT FLEET

Type	Description	Number Available
A	Postwar 4-engine	10
B	Postwar 2-engine	19
C	Prewar 2-engine	25
D	Prewar 4-engine	15

These aircraft have differences in speed, range, payload capacity, and cost characteristics. The assumed routes and expected traffic loads (the distribution of demand will be discussed later) are given in Table 2.

Table 2

TRAFFIC LOAD BY ROUTE

Route	Route Miles[a]	Expected Number of Passengers[b]	Price One-way Ticket ($)
(1) N.Y.-L.A. (1-stop)	2,475	25,000	130
(2) N.Y.-L.A. (2-stop)	2,475	12,000	130
(3) N.Y.-Dallas (0-stop)	1,381	18,000	70
(4) N.Y.-Dallas (1-stop)	1,439	9,000	70
(5) N.Y.-Boston (0-stop)	185	60,000	10

[a] Official Airline Guide, July, 1954, p. 276. The New York-Los Angeles routes are via Chicago and via Chicago and Denver; the stop en route between New York and Dallas is at Memphis.

[b] This is the expected number of full one-way trips per month to be carried on each route. If a passenger gets off en route and is replaced by another passenger, it is counted as one full trip.

Since this paper proposes to illustrate the applicability of a method of solving problems in which several realistic elements are considered, it is assumed that not all aircraft can carry their full loads on all routes and that the obtainable utilization varies from route to route. Specifically, Type B is assumed to be able to operate at only 75 per cent payload on Route 3, and Type D at 80 per cent on Route 1; Type C cannot fly either Route 1 or Route 3, and Type B cannot fly Route 1. Utilization is defined as the average number of hours of useful work performed per month by each aircraft assigned to a particular

route. Utilization of 300 hours per month is assumed on Routes 1 and 2, 285 on Routes 3 and 4, and 240 on Route 5.

The assumed dollar costs per 100 passenger-miles are shown in Table 3. These do not include any capital costs such as those of the aircraft and ground facilities. They represent variable costs such as the cost of gasoline, salaries of the crew, and costs of servicing the aircraft.

A second sort of "cost" is the loss of revenue when not enough aircraft are assigned to the route to meet the passenger demand. In this case, the loss of revenue is the same as the price of a one-way ticket shown in the E row of Table 3.

Table 3

DOLLAR COSTS

Type of Aircraft	Route				
	1 - N.Y. to L.A. 1-stop ($)	2 - N.Y. to L.A. 2-stop ($)	3 - N.Y. to Dallas 0-stop ($)	4 - N.Y. to Dallas 1-stop ($)	5 - N.Y. to Boston 0-stop ($)
	Per 100 Passenger-miles				
1 - A	0.45	0.57	0.45	0.47	0.64
2 - B	-	0.64	0.83	0.63	0.88
3 - C	-	0.92	-	0.93	1.13
4 - D	0.74	0.61	0.59	0.62	0.81
	Per Passenger Turned Away[a]				
5 - E	130 (13)	130 (13)	70 (7)	70 (7)	10 (1)

[a]Figures shown in parentheses are 1000's of dollars lost per 100 passengers turned away. (Throughout this paper, passengers are measured in units of hundreds.)

Based on the speeds, ranges, payload capacities, and turn-around times, passenger-carrying capabilities were determined. The resultant potential number p_{ij} (in hundreds) of passengers that can be flown per month per aircraft of type i on Route j is shown in Table 4; see the staggered upper right figure in each box. By multiplying these numbers by the corresponding costs per 100 passenger-miles given in Table 3 and by the number of miles given in Table 2, the monthly cost per aircraft can also be obtained. This is given in the lower left figure c_{ij} in each box; explicitly, c_{ij} is the cost in thousands of dollars per month per aircraft of type i assigned to the Route j. The revenue losses c_{5j}, in thousands of dollars per 100 passengers not carried, are given in the E row of Table 4; finally, we define $p_{5j} = 1$.* The staggered layout of Table 4 was chosen so as to identify the corresponding data found in Table 5; the latter is the <u>work sheet</u> upon which the entire problem is solved.

The basic problem is that of determining the number of aircraft of each type to assign to each route consistent with aircraft availabilities (Table 1) and of determining how much revenue will be lost due to failure of allocated aircraft to meet passenger demand on various routes (Tables 2 and 3). Since many alternative allocations are possible, our specific objective will be to find that allocation that minimizes total costs, where costs are defined as operating costs plus lost revenues based on the cost factors given in Table 3.

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*This will make it easier to form the passenger-balance or "column" equations (2).

Table 4

PASSENGER-CARRYING CAPABILITIES[a] AND COSTS[b]

Type of Aircraft	Route: 1 - N.Y. to L.A. 1-stop	2 - N.Y. to L.A. 2-stop	3 - N.Y. to Dallas 0-stop	4 - N.Y. to Dallas 1-stop	5 - N.Y to Boston 0-stop
Per Aircraft per Month					
1 - A	$p_{11}=16$ $c_{11}=18$	$p_{12}=15$ $c_{12}=21$	$p_{13}=28$ $c_{13}=18$	$p_{14}=23$ $c_{14}=16$	$p_{15}=81$ $c_{15}=10$
2 - B	*	$p_{22}=10$ $c_{22}=15$	$p_{23}=14$ $c_{23}=16$	$p_{24}=15$ $c_{24}=14$	$p_{25}=57$ $c_{25}=9$
3 - C	*	$p_{32}=5$ $c_{32}=10$	*	$p_{34}=7$ $c_{34}=9$	$p_{35}=29$ $c_{35}=6$
4 - D	$p_{41}=9$ $c_{41}=17$	$p_{42}=11$ $c_{42}=16$	$p_{43}=22$ $c_{43}=17$	$p_{44}=17$ $c_{44}=15$	$p_{45}=55$ $c_{45}=10$
Per 100 Passengers Not Carried (Losses)					
5 - E	$p_{51}=1$ $c_{51}=13$	$p_{52}=1$ $c_{52}=13$	$p_{53}=1$ $c_{53}=7$	$p_{54}=1$ $c_{54}=7$	$p_{55}=1$ $c_{55}=1$

[a]Capabilities p_{ij} are measured in hundreds of passengers.

[b]Costs c_{ij} are measured in thousands of dollars.

This fixed-demand model may be formulated mathematically as a linear programming problem. Let x_{ij} denote the unknown number of aircraft of the i^{th} type assigned to j^{th} route, where $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n-1$. If x_{in} denotes the number of surplus or unallocated aircraft of the i^{th} type, then Eq. (1) below states that the sum of allocated and unallocated aircraft of the i^{th} type accounts for the total number a_i of available aircraft of this type. If $x_{m+1,j}$ denotes the number of passengers in hundreds turned away from the j^{th} route per month, then Eq. (2) states that the sum of the passenger-carrying capabilities of all aircraft allocated to the j^{th} route, plus the unsatisfied demand, accounts for the total demand d_j on the route. Relation (3) states that all unknown quantities x_{ij} must be either positive or zero. Finally, if c_{in} $(i = 1, 2, \ldots, m)$ is the monthly cost of maintaining an aircraft of the i^{th} type when not in use, then the total cost z is the sum of all the individual operating costs plus the revenue lost by unsatisfied demands $c_{m+1,j}\, x_{m+1,j}$, as given in Eq. (4).

FIXED-DEMAND MODEL

Find numbers x_{ij}, and the minimum value of z, satisfying the following conditions.

(1) Row Sums: $x_{i1} + x_{i2} + \ldots + x_{in} = a_i$

$(i = 1, 2, \ldots, m)$,

(2) Column Sums: $p_{1j}x_{1j} + p_{2j}x_{2j} + \ldots + p_{mj}x_{mj} = d_j$

$(j = 1, 2, \ldots, n-1)$,

(3) $x_{ij} \geq 0$,

(4) $\sum_{i=1}^{m+1} \sum_{j=1}^{n} c_{ij}x_{ij} = z$.

Any set of assignments x_{ij} satisfying Eqs. (1), (2), and (3) is termed a <u>feasible solution</u>, and a feasible choice that minimizes the total cost z of the assignment is called an <u>optimal (feasible) solution</u>.

Table 5 shows the optimal assignment of aircraft to routes, based on fixed demand, as developed in the earlier study. The values assigned to the unknowns x_{ij} appear underlined in the upper left of each box unless $x_{ij} = 0$ in which case it is omitted; the entire layout takes the form:

$\underline{x_{ij}}$	
	p_{ij}
c_{ij}	

The sums by rows of the x_{ij} entries in Table 5 equated to availabilities yield Eqs. (1). The sums by columns of the x_{ij} weighted by corresponding values of p_{ij} equated to demands yield Eqs. (2); the x_{ij} weighted by corresponding c_{ij} and summed over the entire table yield Eq. (4). As noted earlier, Table 5 is actually the work sheet upon which the entire problem is solved. Later we shall discuss a revision of this work sheet for solving problems with variable demand. All figures in the table, except for the upper left entries x_{ij} and values of the so-called "implicit prices" u_i and v_j shown in the margins, are constants that do not change during the course of computation. The values of the variables x_{ij}, u_i, and v_j, however, will change during the course of successive iterations of the simplex method as adapted for this problem. For this reason it is customary to cover the work sheet with clear acetate and to enter the variable information with a grease pencil so that the marks can be easily erased; alternatively, a blackboard or semitransparent tissue-paper overlays can be used. The detailed rules for obtaining the optimal solution shown are given in [1] and will not be repeated here. Instead, a more general set of rules for the uncertain-demand case will be given; these, of course, could be used in particular for the expected-demand case.

In the following outline we have a convenient summary that serves to identify and define the numerical data entered in Table 5 and to give the test for optimality.

Table 5

OPTIMAL ASSIGNMENT FOR FIXED DEMAND
Operating Costs and Lost Revenues = $1,000,000

Type of Air-craft	Route						Air-craft Avail-able	Im-plicit Prices u_i
	(1) N. Y. to L. A. 1-stop	(2) N. Y. to L. A. 2-stop	(3) N. Y. to Dallas 0-stop	(4) N. Y. to Dallas 1-stop	(5) N. Y. to Boston 0-stop	(6) Sur-plus Air-craft		
(1) A	10 16 18	15 21	28 18	23 16	81 10	0 0	10=a_1	-171
(2) B	**	8 10 15	5 14 16	6 15 14	57 9	0 0	19=a_2	- 51
(3) C	**	7.8 5 10	**	7 9	17.2 29 6	0 0	25=a_3	- 23
(4) D	10 9 17	11 16	5 22 17	17 15	55 10	0 0	15=a_4	- 89
(5) E Deficit	1 13	1 13	1 7	1 7	100 1 1	0 0	**	0
Demand d_j	250	120	180	90	600	**		
Im-plicit Prices v_j	11.8	6.6	4.8	4.33	1	0		

SUMMARY	
Constants:	a_i = number of available aircraft of type i d_j = expected passenger demand in 100's per month on Route j p_{ij} = passenger-carrying capability in 100's per month per aircraft of type i assigned to Route j ($p_{m+1,j}$ = 1 by definition) c_{ij} = costs in 1000's of dollars per month per aircraft of type i assigned to Route j ($c_{m+1,j}$ is per 100 passengers turned away)
x_{ij} Entries:	x_{ij} = number of aircraft of type i assigned to Route j ($x_{m+1,j}$ is 100's of passengers turned away) Omitted x_{ij} Entries: x_{ij} = 0 if upper left entry in box is missing
Implicit Prices:	u_i and v_j are determined such that $u_i + p_{ij}v_j = c_{ij}$ for (i,j) boxes with $x_{ij} > 0$—i.e., with underlined entries. Note: $u_{m+1} = v_n = 0$
Test for Optimality:	Solution is optimal if, for <u>all</u> (i,j), the relation $u_i + p_{ij}v_j \leq c_{ij}$ holds

3. EXTENSION OF EXAMPLE TO UNCERTAIN DEMAND

Up to this point the problem is identical with that described and solved in our previous paper. Now, to introduce the element of uncertain demand, we assume not a known (expected) demand on each route but a known frequency distribution of demand. The assumed frequency distributions are shown in Table 6. Thus on Route 5 (N.Y. to L.A. - 2-stop) either 5,000 or 15,000 passengers will want transportation during the month, with probabilities 30 or 70 per cent respectively. The assumed traffic distributions are, of course, hypothetical to illustrate our method. The demand distributions on the five routes vary over wide ranges and have different characteristics; Route 1 is flat, Route 2 is U-shaped, Routes 3, 4, and 5 are unimodular but have differing degrees of concentration about the mode. Route 4 has a distribution with a very long tail that may reflect a realistic traffic situation.

Table 6

ASSUMED DISTRIBUTION OF PASSENGER DEMAND
λ_{hj} = Probability of Demand d_{hj}

Route	Passengers (in hundreds)	Approx. Mean (in hundreds)	Probability of Passenger Demand	Probability of Equaling or Exceeding Demand
1 – A	200 = d_{11}	250	0.2 = λ_{11}	1.0 = γ_{11}
	220 = d_{21}		0.05 = λ_{21}	0.8 = γ_{21}
	250 = d_{31}		0.35 = λ_{31}	0.75 = γ_{31}
	270 = d_{41}		0.2 = λ_{41}	0.4 = γ_{41}
	300 = d_{51}		0.2 = λ_{51}	0.2 = γ_{51}
2 – B	50 = d_{12}	120	0.3 = λ_{12}	1.0 = γ_{12}
	150 = d_{22}		0.7 = λ_{22}	0.7 = γ_{22}
3 – C	140 = d_{13}	180	0.1 = λ_{13}	1.0 = γ_{13}
	160 = d_{23}		0.2 = λ_{23}	0.9 = γ_{23}
	180 = d_{33}		0.4 = λ_{33}	0.7 = γ_{33}
	200 = d_{43}		0.2 = λ_{43}	0.3 = γ_{43}
	220 = d_{53}		0.1 = λ_{53}	0.1 = γ_{53}
4 – D	10 = d_{14}	90	0.2 = λ_{14}	1.0 = γ_{14}
	50 = d_{24}		0.2 = λ_{24}	0.8 = γ_{24}
	80 = d_{34}		0.3 = λ_{34}	0.6 = γ_{34}
	100 = d_{44}		0.2 = λ_{44}	0.3 = γ_{44}
	340 = d_{54}		0.1 = λ_{54}	0.1 = γ_{54}
5 – E	580 = d_{15}	600	0.1 = λ_{15}	1.0 = γ_{15}
	600 = d_{25}		0.8 = λ_{25}	0.9 = γ_{25}
	620 = d_{35}		0.1 = λ_{35}	0.1 = γ_{35}

To illustrate the essential character of the linear-programming problem for the case of uncertain demand, let us focus our attention on a single route—say, Route 1—with probability distribution of demand as given in Table 6. Let us suppose that aircraft assigned to Route 1 are capable of hauling 100 Y_1 passengers. The first 200 units (in hundreds of passengers) of this capability are certain to be used, and revenues from this source (negative costs) will be $13 = k_1$ units (in thousands of dollars) per unit of capability. The next 20 units of this capability will be used with probability $\gamma_{21} = 0.8$. Indeed, 80 per cent of the time the demand will be 220 units or greater, while 20 per cent of the time it will be 200 units; hence, the expected revenues per unit from this increment of capability is $0.8 \times 13 = 10.4$, or $10.4 = k_1\gamma_{21}$ units. On the third increment of 30 units (22,001 to 25,000 seats) the expected revenue is $0.75 \times 13 = 9.8 = k_1\gamma_{31}$ units per unit of capability since there is a 25 per cent chance that none of these units of capability will be used and 75 per cent that all will be used. For the fourth increment of 20 units (25,001 to 27,000 seats) of capability the expected revenue is $0.4 \times 13 = 5.2 = k_1\gamma_{41}$ units per unit of capability, while for the fifth increment of 30 units (27,001 to 30,000 seats) it is $0.2 \times 13 = 26 = k_1\gamma_{51}$ units per unit. For the sixth increment, which is the number of units assigned above the 30,000 seat mark, the expected revenue is $0.0 \times 13 = 0$ per unit since it is certain that none of these units of capability can be used. It is clear that no assignments above 30,000 seats are worthwhile, and hence the last increment can be omitted.

The index h = 1, 2, 3, 4, 5 will be used to denote the 1st, 2nd, ..., 5th increment of demand.

The number of assigned units in each increment, however, can be viewed as an unknown that depends on the total (passenger-hauling) capability assigned to Route j = 1. Thus if the total assigned is $Y_1 = 210$ units of capability then the part of this total belonging to the first increment, denoted by y_{11}, is $y_{11} = 200$ and the part belonging to the second increment, denoted by y_{21}, is $y_{21} = 10$; the amounts in the higher increments are $y_{hi} = 0$ for $i = 3, 4, 5$. To review, the passenger-carrying capability Y_j is determined by the number of aircraft assigned to Route j, so that

(5) $$Y_j = p_{1j}\, x_{1j} + p_{2j}\, x_{2j} + p_{3j}\, x_{3j} + p_{4j}\, x_{4j}\,.$$

On the other hand, Y_j itself breaks down into five increments

(6) $$Y_j = y_{1j} + y_{2j} + y_{3j} + y_{4j} + y_{5j}$$

for Routes j = 1, 3, 4, and correspondingly fewer for j = 2, 5. Regardless of the total Y_j, the amount y_{hj} belonging to each increment is bounded by the total size b_{hj} of that increment; the latter, however, is simply the change in demand level, so that

(7) $$\begin{aligned} 0 \le y_{1j} &\le d_{1j} &&= b_{1j}, \\ 0 \le y_{2j} &\le d_{2j} - d_{1j} &&= b_{2j}, \\ 0 \le y_{3j} &\le d_{3j} - d_{2j} &&= b_{3j}, \\ 0 \le y_{4j} &\le d_{4j} - d_{3j} &&= b_{4j}, \\ 0 \le y_{5j} &\le d_{5j} - d_{4j} &&= b_{5j}. \end{aligned}$$

The total expected revenue from Route j is, therefore,

$$(8)\qquad k_j(\gamma_{1j}\, y_{1j} + \gamma_{2j}\, y_{2j} + \dots + \gamma_{5j}\, y_{5j}),$$

where k_j is revenue (in thousands of dollars) per 100 passengers carried on Route j, and where, as seen in Table 6,

$$(9)\qquad \begin{aligned} 1 = \gamma_{1j} &= \lambda_{1j} + \lambda_{2j} + \lambda_{3j} + \lambda_{4j} + \lambda_{5j}, \\ \gamma_{2j} &= \lambda_{2j} + \lambda_{3j} + \lambda_{4j} + \lambda_{5j}, \\ \gamma_{3j} &= \lambda_{3j} + \lambda_{4j} + \lambda_{5j}, \\ \gamma_{4j} &= \lambda_{4j} + \lambda_{5j}, \\ \gamma_{5j} &= \lambda_{5j}. \end{aligned}$$

For example, the total expected revenue for Route 1 is

$$(10)\qquad 13(1.0y_{11} + .8y_{12} + .75y_{13} + .4y_{14} + .2y_{15}).$$

The most important fact to note about the linear form (10) is the decrease in the successive values of the coefficients γ_{hj}. Moreover, this will always be the case whatever the distribution of demand since the probability of equaling or exceeding a given demand level d_{hj} decreases with increasing values of demand.

Suppose now that y_{11}, y_{21}, ... are treated as unknown variables in a linear-programming problem subject only to (6) and (7), where the objective is to maximize revenues. Let us suppose further that Y_1 is fixed. It is clear, since the coefficient of y_{11} is largest in the maximizing form (8), that y_{11} will be chosen as large as possible consistent with (6) and (7); for the chosen value y_{11}, the next increment y_{21} will be chosen as large as possible consistent with (6) and (7), etc.

Thus, we need only specify y_{h1} by restrictions (6) and (7), because when the maximum is reached the values of the variables $y_{11}, y_{21}, \ldots$ are precisely the incremental values (6) associated with Y_1. Even if passenger capability Y_1 is not fixed, as in the case about to be considered, it should be noted that whatever be the value of Y_1 the values of $y_{11}, y_{21} \ldots$ that minimize an over-all cost form such as (14) below must maximize (8) for $j = 1$, so that the incremental values of Y_1 will be generated by $y_{11}, y_{21}, \ldots$.

The linear-programming problem in the case of uncertain demand becomes:

UNCERTAIN DEMAND MODEL

Find numbers x_{ij} and y_{hj}, and the minimum value of z, satisfying the following conditions.

(11) Row Sums: $x_{i1} + x_{i2} + \ldots + x_{in} = a_i \quad (i = 1, 2, \ldots, m)$

(12) Column Sums: $p_{1j}x_{1j} + p_{2j}x_{2j} + \ldots + p_{mj}x_{nj}$

$$= y_{1j} + y_{2j} + \ldots + y_{rj} \quad (j = 1, 2, \ldots, n-1)$$

(13) $x_{ij} \geq 0, \quad (i = 1, \ldots, m;\ j = 1, \ldots, n)$

$$0 \leq y_{hj} \leq b_{hj} \quad (h = 1, \ldots, r;\ j = 1, \ldots, n-1)$$

(14) Expected Costs: $$z = \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij}x_{ij} + \left[R_0 - \sum_{j=1}^{n-1} k_j \sum_{h=1}^{r} \gamma_{hj}y_{hj}\right]$$

Here R_0 is the value that expected revenue would be if sufficient seats were supplied for all customers. Thus expected costs are defined as total outlays (first term) plus the expected loss of revenue due to shortage of seats (last two terms).

For the problem at hand, the bounds b_{hj} and the expected revenues γ_{hj} per unit for the "incremental variables" y_{hj} can be computed from the probability distributions of Table 6 via (7) and (9).

The numerical values of the constants for the stochastic case are shown in Table 7.

Table 7

INCREMENTAL BOUNDS b_{hj} AND EXPECTED REVENUES $k_j \gamma_{hj}$
PER UNIT OF ASSIGNED PASSENGER-CARRYING CAPABILITY

Increment h	Route 1		Route 2		Route 3		Route 4		Route 5	
	b_{h1}	$k_1\gamma_{h1}$	b_{h2}	$k_2\gamma_{h2}$	b_{h3}	$k_3\gamma_{h3}$	b_{h4}	$k_4\gamma_{h4}$	b_{h5}	$k_5\gamma_{h5}$
1	200	13	50	13	140	7	10	7	580	1
2	20	10.4	100	9.1	20	6.3	40	5.6	20	0.9
3	30	9.8	**		20	4.9	30	4.2	20	0.1
4	20	5.2	**		20	2.1	20	2.1	**	
5	30	2.6	**		20	0.7	240	0.7	**	

**Only two increments for Route 2 and three increments for Route 5 are needed to describe the distribution of demand.

4. RULES FOR COMPUTATION

The work sheet for determining the optimal assignment under uncertain demand is shown in Table 9. To form the new row equations (11), the x_{ij} entries are summed to yield the a_i values given in the Aircraft Available column. To form the column equations (12), the x_{ij} entries are multiplied by the corresponding p_{ij}, the y_{hj} by -1, and summed down to yield zero.

Step 1. To initiate the computation any set of non-negative values may be assigned to the unknowns x_{ij} and y_{hj} provided they satisfy the equations and thereby constitute a feasible solution.

Step 2. Circle any m + n of the x_{ij} and y_{hj} entries, where m + n is the total number of row and column equations. These circles can be arbitrarily selected except that they must have the property that if the fixed values assigned to the other non-circled variables and the constant terms were arbitrarily changed to other values then the circled variables would be determined uniquely in terms of the latter. Such a circled set of variables is called a basic set of variables; the array of coefficients associated with this set in the equations (11) and (12) is referred to as the basis in the theory of the simplex method [4].

Note: One simple way of selecting a basic set is shown in Table 10. One x_{ij} entry is arbitrarily selected and circled in each row corresponding to a row equation, and one y_{hj} is arbitrarily selected and circled in each column corresponding to a column equation. In general, it is suggested that entries be circled that appear to have a chance of having a positive value

in an optimum solution; for y_{hj} values, the last entry in the column that appears likely to be positive in an optimum solution should be circled.

Step 3. For (i,j) and (h,j) combinations corresponding to circled entries, compute implicit prices u_i and v_j associated with equations by determining values of u_i and v_j satisfying the equations

(15) $u_i + p_{ij}v_j = c_{ij}$ (x_{ij} circled),

(16) $0 + (-1)v_j = -k_j\gamma_{hj}$ (y_{hj} circled).

There are always m + n equations (15) and (16) in m + n unknowns u_i and v_j that can be shown easily to have a unique solution [4]. They can be solved by inspection, for it can be shown that the system either is completely triangular or at worst contains sub-systems—some triangular and some triangular if one unknown is specified.*

Step 4. For each box corresponding to x_{ij} or y_{hj}, compute

(17) $\delta_{ij} = (u_i + p_{ij}v_j) - c_{ij}$ (for x_{ij} box),

(18) $\delta'_{hj} = (0 - v_j) - (-k_j\gamma_{hj})$ (for y_{hj} box).

*This is the analogue—for the "generalized" transportation problem (1), (2), (3), (4)—of the well-known theorem for the standard transportation problem that all bases are triangular. Its proof is similar.

In practice, one of the δ_{ij} or δ'_{hj} is recorded; the others are computed and compared with it, and the largest in absolute value is used. It can be shown [4] that if the x_{ij} or y_{hj} value associated with a noncircled entry is changed to

$$x_{ij} \pm \theta \qquad \text{or} \qquad y_{hj} \pm \theta \qquad (\theta \geq 0),$$

the other noncircled variables remaining invariant, and the circled variables adjusted, then the expected costs z will change to z', where

$$z' = z \mp \theta\delta_{ij} \qquad \text{or} \qquad z' = z \mp \theta\delta'_{hj} \ .$$

Thus it pays to <u>increase</u> x_{ij} or y_{hj} if δ_{ij} or $\delta'_{hj} > 0$, unless y_{hj} is equal to its upper bound b_{hj}, in which case no increase in y_{hj} is allowed; also it pays to <u>decrease</u> x_{ij} or y_{hj} if δ_{ij} or $\delta'_{hj} < 0$ unless $x_{ij} = 0$ or $y_{hj} = 0$, in which case no decrease is allowed.

<u>Test for Optimality</u>: According to the theory of the simplex method [3] if the <u>noncircled</u> variables satisfy the following conditions:

(a) each one is at either its upper or its lower bound value,

(b) the corresponding δ_{ij} or δ'_{hj} is less than or equal to 0, if it is at its lower bound value, and

(c) the corresponding δ_{ij} or δ'_{hj} is greater than or equal to 0 if it is at its upper bound value,

then the solution is optimal and the algorithm terminates. Otherwise there are δ_{ij} or δ'_{hj} for which a decrease or increase

(depending on whether the sign is negative or positive) in the corresponding variable is allowed; let the largest among them in absolute value be denoted by δ_{rs} or δ'_{rs}.

Step 5. Leaving all noncircled entries fixed except for the value of the variable corresponding to the (r,s) box determined in Step 4, modify the value of x_{rs} (or y_{rs}) to

$$x_{rs} + \theta \text{ (or } y_{rs} + \theta\text{) if } \delta_{rs} > 0 \text{ (or } \delta'_{rs} > 0\text{)}$$

$$x_{rs} - \theta \text{ (or } y_{rs} - \theta\text{) if } \delta_{rs} < 0 \text{ (or } \delta'_{rs} < 0\text{)},$$

where $\theta \geq 0$ is unknown, and recompute the values of circled variables as linear functions of θ. Choose the value of $\theta = \theta^*$ as the largest value possible consistent with keeping all basic (circled) variables [whose values now depend on θ] between their upper and lower bounds; in the next cycle correct the values of the circled variables on the assumption that $\theta = \theta^*$.

Also, if at the value $\theta = \theta^*$ one (or more) of the circled variables attains its upper or lower bound, in the next cycle drop just one such variable from the basic set and circle the variable x_{rs} instead. Should it happen that it is x_{rs} that attains its upper or lower bound at $\theta = \theta^*$, the set of circled variables is the same as before; their values, however, are changed to allow x_{rs} to be fixed at its new bound.

Start the next cycle of the iterative procedure by returning to Step 3.

5. NUMERICAL SOLUTION OF THE ROUTING PROBLEM

For our starting solution we used for values of the x_{ij} the best solution of the earlier study, assuming fixed demands equal to the expected values of the distribution.* These are shown in Table 10. These x_{ij} will meet the expected demands, so that $Y_j = b_j$ except for Route 5; there is a deficit of 100 for this route, and by (5) we have $Y_5 = 500$. These Y_j are broken down into the successive incremental values shown below the double line in Table 10; see Eq. (6).

Next, one of the variables in each row is circled. In the example, the selected variables are x_{11}, x_{22}, x_{35}, and x_{43}; each appears likely to be in an optimal solution, though x_{41} may turn out to be a better choice than x_{43}. Next, the last positive entry in each column is circled; in the example, these are the variables y_{31}, y_{22}, y_{33}, y_{44}, and y_{15}. In all, there are $m + n$ circled variables (9 in the example). The implicit values must satisfy the $m + n$, or 9, equations:

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*In the humorous parody by Paul Gunther, entitled "Use of Linear Programming in Capital Budgeting," Journal of the Operations Research Society of America, May, 1955, Mrs. Efficiency wondered why Mr. O. R. did not start out with a good guess. It will be noted that in this paper we have followed Mrs. Efficiency's suggestion and have started with a guess at the final solution rather than going through the customary use of artificial variables and a Phase 1 of the simplex process.

$$u_1 + p_{11}v_1 = c_{11} \qquad (p_{11}=16,\ c_{11}=18),$$
$$u_2 + p_{22}v_2 = c_{22} \qquad (p_{22}=10,\ c_{22}=15),$$
$$u_3 + p_{35}v_5 = c_{35} \qquad (p_{35}=29,\ c_{35}=6),$$
$$u_4 + p_{43}v_3 = c_{43} \qquad (p_{43}=22,\ c_{43}=17),$$
$$0 + (-1)v_1 = -k_1\gamma_{31} \qquad (k_1\gamma_{31}=9.8),$$
$$0 + (-1)v_2 = -k_2\gamma_{22} \qquad (k_2\gamma_{22}=9.1),$$
$$0 + (-1)v_3 = -k_3\gamma_{33} \qquad (k_3\gamma_{33}=4.9),$$
$$0 + (-1)v_4 = -k_4\gamma_{44} \qquad (k_4\gamma_{44}=2.1),$$
$$0 + (-1)v_5 = -k_5\gamma_{15} \qquad (k_5\gamma_{15}=1.0).$$

This permits the computation of δ_{ij} and δ'_{hj}; see (17) and (18). As a check, $\delta_{ij} = 0$ and $\delta'_{hj} = 0$ for (i,j) and (h,j) corresponding to circled variables. The δ_{ij} or δ'_{hj} of largest absolute value is

$$\delta_{24} = [-76 + 15\ (2.1)]-14 = -58.5;$$

hence a decrease in the variable x_{24} with adjustments of the circled variables will result in a decrease in the expected costs by an amount of 58.5 units per unit decrease in x_{24}. If $x_{24} = 6$ is changed to $x_{24} = 6 - \theta$, then, in order to satisfy the column 4 equation, the circled variable $y_{44} = 10$ must be modified to $y_{44} = 10 - 15\theta$ (all other variables in column 4 are fixed). Also, to satisfy the row 2 equation, $x_{22} = 8$ must be modified to $x_{22} = 8 + \theta$; this in turn causes $y_{22} = 70$ to be changed to $y_{22} = 70 + 10\theta$ in order to satisfy the column 2 equation. The largest value of θ is $\theta^* = 10/15$, at which value $y_{44} = 0$.

The numerical values of the variables appearing in Table 11 are obtained from those of Table 10 by setting $\theta = \theta^* = 10/15$. The variable x_{24} becomes a new circled variable in place of y_{44}, which hit its lower bound, zero; the other variables to be circled remain the same as in Table 10. Computing the new set of implicit prices, we see that the δ_{ij} of largest absolute value that can be increased or decreased (according to the sign of δ_{ij}) is δ_{23} = 25.4. Changing x_{23} to [illegible] $- \theta$ requires that the variables x_{22}, y_{22}, and y_{33} be modified as shown in Table 11. The maximum value of θ is $\theta = \theta^* = 20/14$, at which value we have $y_{33} = 0$. The new solution, in which x_{23} replaces y_{33} as a circled variable, is given in Table 12, where the decrease in the noncircled variable x_{41} causes changes in the variables x_{43}, x_{22}, x_{23}, y_{31}, and y_{22}. The largest value of θ is 9/10, at which value y_{22} hits its <u>upper bound</u> b_{22} = 100.

In the passage from Table 13 to Table 14 we have taken a "double" step. The maximum increase is $\theta = 80/20$, at which point y_{15} hits its upper bound b_{15} = [illegible]80. It is easy to see that if next the incremental variable y_{25} is increased then δ_{32} associated with x_{32} should be changed to $\delta_{32} + 20\,(\gamma_{15} - \gamma_{25})k_5 = -4.5 +$ $2\cdot(1.0 - .9)$ = [illegible]; therefore, it is economical to increase y_{25} as well as y_{15}. However, it can be shown that the sign of δ_{32} would become positive if the next increment, y_{35}, were considered. The maximum value of $\theta = \theta^*$ is 100/20.

In the passage from Table 14 to [illegible], it will be noted that the variable y_{33}, which had been dropped earlier, is again brought into the solution. The maximum value of θ is 22/20, at which

value y_{33} reaches its upper bound, so that the new solution, given in Table 15, has the same set of circled variables and hence the same implicit values as those in Table 14. Moreover, the solution is optimal since all noncircled variables are either at their upper or lower bounds—those at upper bounds have corresponding $\delta_{ij} \geq 0$ and those at lower bounds have $\delta_{ij} \leq 0$.

In comparing this solution (Table 15) with the optimal solution for the fixed-demand case (Table 5), it is interesting to note that the chief difference appears to be a general tendency, in the case of distribution with sharp peaks, to shift the total seats made available on route to a mode of the distribution rather than to the mean of the distribution. The total seats made available on routes with flat distributions of demand, on the other hand, appear to be at the highest level attainable with the residual passenger-carrying potential.

To compute the expected costs of the various solutions, the first step (see Eq. (14)) is to determine what the expected revenues R_0 would be if sufficient seating capacity were furnished at all times to supply all passengers that show. From Table 2 it is easy to see that

$$R_0 = 13(250) + 13(120) + 7(180) + 7(90) + 1(600) = 7300,$$

so that the expected revenue would be \$7,300,000.

Table 8

COMPARATIVE COSTS OF VARIOUS SOLUTIONS

Table	Expected Revenues For Seats Supplied (1)	Expected Lost Revenues* (2)	Operating Costs (3)	Net Expected Cost (Thousands) (2) + (3)
10	-6534	766	900	1,666
11	-6574	726	901	1,627
12	-6607	693	901	1,594
13	-6638	662	899	1,561
14	-6641	659	883	1,542
15	-6659	641	883	1,542

*Data in column (2) are obtained by subtracting the expected revenues for seats supplied, column (1), from R_0 = 7300 = the expected revenues if an unlimited number of seats were supplied.

It is seen that the solution presented in the earlier paper [1], assuming demands to be exactly equal to their expected values, has a net expected cost of $1,666,000. [It is interesting to note that if the demands were fixed and equal to their expected values, the costs would be only $1,000,000 (see Table 5). The 67 per cent increase in net cost for the variable-demand case is due to 13,400 additional passengers (on the average) being turned away because of the distributions of demand assumed in Table 6.] The successive improvements in the solution given in Tables 10 to 15 reduce the net expected costs from $1,666,000 to $1,524,000 for the optimal solution.

In the illustration the best solution obtained by pretending that demands are fixed at these expected values has a 9 per cent higher expected cost than that for the best solution obtained by using the assumed distributions of demand. It is also seen that very little additional computational effort was required to take account of this uncertainty of demand.

Table 9

WORK SHEET FOR DETERMINING OPTIMAL ASSIGNMENT UNDER UNCERTAIN DEMAND

Type of Aircraft	Route (1) N. Y. to L. A. 1-stop	(2) N. Y. to L. A. 2-stop	(3) N. Y. to Dallas 0-stop	(4) N. Y. to Dallas 1-stop	(5) N. Y. to Boston 0-stop	(6) Surplus Aircraft	Aircraft Available	Implicit Prices u_i
(1) A	x_{11} $p_{11}=16$ $c_{11}=18$	x_{12} 15 21	x_{13} 28 18	x_{14} 23 15	x_{15} 81 10	x_{16} 0 0	10	u_1
(2) B	***	x_{22} 10 15	x_{23} 14 16	x_{24} 15 14	x_{25} 57 9	x_{26} 0 0	19	u_2
(3) C	***	x_{32} 5 10	***	x_{34} 7 9	x_{35} 29 6	x_{36} 0 0	25	u_3
(4) D	x_{41} 9 17	x_{42} 11 16	x_{43} 22 17	x_{44} 17 15	x_{45} 55 10	x_{46} 0 0	15	u_4
Increment (1)	$y_{11}\leq 200$ −1 −13	$y_{12}\leq 50$ −1 −13	$y_{13}\leq 140$ −1 −7	$y_{14}\leq 10$ −1 −7	$y_{15}\leq 580$ −1 −1	***	***	0
(2)	$y_{21}\leq 20$ −1 −10.4	$y_{22}\leq 100$ −1 −9.1	$y_{23}\leq 20$ −1 −6.3	$y_{24}\leq 40$ −1 −5.6	$y_{25}\leq 20$ −1 −.9	***	***	0
(3)	$y_{31}\leq 30$ −1 −9.8	***	$y_{33}\leq 20$ −1 −4.9	$y_{34}\leq 30$ −1 −4.2	$y_{35}\leq 20$ −1 −.1	***	***	0
(4)	$y_{41}\leq 20$ −1 −5.2	***	$y_{43}\leq 20$ −1 −2.1	$y_{44}\leq 20$ −1 −2.1	***	***	***	0
	$y_{51}\leq 30$ −1 −2.6	***	$y_{53}\leq 20$ −1 −.7	$y_{54}\leq 240$ −1 −.7	***	***	***	0
Net	0	0	0	0	0	***	***	***
Implicit Prices v_j	v_1	v_2	v_3	v_4	v_5	0	***	***

*** Box not used because corresponding row or column has no equation, or else because aircraft type cannot fly required range, or fewer increments are needed to describe the distribution of demand on the route.

Table 10 – Cycle 0

WORK SHEET FOR DETERMINING OPTIMAL ASSIGNMENT UNDER UNCERTAIN DEMAND

$\delta_{24} = 58.4$, $\theta = 10/15$, Expected Cost = $1,660,000

Type of Aircraft	Route (1) N. Y. to L. A. 1-stop	(2) N. Y. to L. A. 2-stop	(3) N. Y. to Dallas 0-stop	(4) N. Y. to Dallas 1-stop	(5) N. Y. to Boston 0-stop	(6) Surplus Aircraft	Aircraft Available	Implicit Prices u_i
(1) A	10 16 18	15 21	28 18	23 16	81 10	0 0	10	−139
(2) B	***	8+θ 10 15	5 14 16	6−θ 15 14	57 9	0 0	19	− 76
(3) C	***	7.8 5 10	***	7 9	17.2 29 6	0 0	25	− 23
(4) D	10 9 17	11 16	5 22 17	17 15	55 10	0 0	15	− 91
Increment (1)	200 −1 −13	50 −1 −13	140 −1 −7	10 −1 −7	500 −1 −1	***	***	0
(2)	20 −1 −10.4	70+10θ −1 −9.1	20 −1 −6.3	40 −1 −5.6	−1 −.9	***	***	0
(3)	50 −1 −9.8	***	20 −1 −4.9	30 −1 −4.2	−1 −.1	***	***	0
(4)	−1 −5.2	***	−1 −2.1	10−15θ −1 −2.1	***	***	***	0
(5)	−1 −2.6	***	−1 −.7	−1 −.7	***	***	***	0
Net	0	0	0	0	0	***	***	***
Implicit Prices v_j	9.8	9.1	4.9	2.1	1	0	***	***

Table 11 - Cycle 1

WORK SHEET FOR DETERMINING OPTIMAL ASSIGNMENT UNDER UNCERTAIN DEMAND

δ_{23} = 23.4, θ = 20/14, Expected Cost = $1,627,000

Type of Aircraft	Route (1) N. Y. to L. A. 1-stop	(2) N. Y. to L. A. 2-stop	(3) N. Y. to Dallas 0-stop	(4) N. Y. to Dallas 1-stop	(5) N. Y. to Boston 0-stop	(6) Surplus Aircraft	Aircraft Available	Implicit Prices u_i
(1) A	10 16 18	15 21	28 18	23 16	81 10	0 0	10	−139
(2) B	***	8.7+θ 10 15	5−θ 14 16	5.3 15 14	57 9	0 0	19	−76
(3) C	***	5 10	***	7 3	17.2 29 6	0 0	25	−23
(4) D	10 9 17	11 16	5 22 17	17 15	55 10	0 0	15	−91
Increment (1)	200 −1 −13	50 −1 −13	140 −1 −7	10 −1 −7	500 −1 −1	***	***	0
(2)	20 −1 −10.4	77+100θ −1 −9.1	20 −1 −6.3	40 −1 −[illegible]	−1 −.9	***	***	0
(3)	30 −1 −9.8	***	20−140θ −1 −4.9	30 −1 −4.2	−1 −.1	***	***	0
(4)	−1 −[illegible].2	***	−1 −2.1	−1 −2.1	***	***	***	0
(5)	−1 −2.6	***	−1 − .7	−1 − .7	***	***	***	0
Net	0	0	0	0	0	***	***	***
Implicit Prices v_j	9.8	9.1	4.9	6	1	0	***	***

Table 12 – Cycle 2

WORK SHEET FOR DETERMINING OPTIMAL ASSIGNMENT UNDER UNCERTAIN DEMAND

$\varepsilon_{41} = -56.8$, $\theta = 9/16$. Expected Cost = $1,594,000

Type of Aircraft	Route (1) N. Y. to L. A 1-stop	(2) N. Y. to L. A. 2-stop	(3) N. Y. to Dallas 0-stop	(4) N. Y. to Dallas 1-stop	(5) N. Y. to Boston 0-stop	(6) Surplus Aircraft	Aircraft Available	Implicit Prices u_i
(1) A	10 16 18	15 21	28 18	23 16	81 10	0 0	10	-139
(2) B	***	10.1 + 1.6θ 10 15	3.6 - 1.6θ 14 16	5.3 15 14	57 9	0 0	19	- 76
(3) C	***	7.8 5 10	***	7 9	17.2 29 6	0 0	25	-23
(4) D	10-θ 9 17	11 16	5+θ 22 17	17 15	55 10	0 0	15	-128
Increment (1)	200 -1 -13	50 -1 -13	140 -1 -7	10 -1 -7	500 -1 -1	***	***	0
(2)	20 -1 -10.4	91 +16θ -1 -9.1	20 -1 -6.3	40 -1 -5.6	-1 -.9	***	***	0
(3)	30-9θ -1 -9.8	***	-1 -4.9	30 -1 -4.2	-1 -.1	***	***	0
(4)	-1 -5.2	***	-1 -2.1	-1 -2.1	***	***	***	0
(5)	-1 -2.6	***	-1 -.7	-1 -.7	***	***	***	0
Net	0	0	0	0	0	***	***	***
Implicit Prices v_j	9.8	9.1	6.6	6	1	0	***	***

Table 13 - Cycle 3

WORK SHEET FOR DETERMINING OPTIMAL ASSIGNMENT UNDER UNCERTAIN DEMAND

$\delta_{32} = 5.5$, $\theta = 100/29 = 3.45$, Expected Cost = $1,561,000

Type of Air-craft	Route (1) N. Y. to L. A. 1-stop	(2) N. Y. to L. A. 2-stop	(3) N. Y. to Dallas 0-stop	(4) N. Y. to Dallas 1-stop	(5) N. Y. to Boston 0-stop	(6) Sur-plus Air-craft	Air-craft Avail-able	Im-plicit Prices u_i
(1) A	10 16 18	15 21	28 18	25 15	81 10	0 0	10	-139
(2) B	...	11+.30 10 [illegible]	2.[illegible]-.[illegible]θ 14 16	[illegible].3 1[illegible] 14	[illegible]7 9	0 0	19	-40
(3) C	...	7.8-θ 5 10	...	7 9	17.2+θ 29 6	0 0	25	-23
(4) D	9.4-.3θ 9 17	11 16	5.6+.3θ 22 17	17 15	55 10	0 0	15	-71
Incre-ment (1)	200 -1 -13	50 -1 -13	140 -1 -7	10 -1 -7	500+29θ -1 -1	...	...	0
(2)	20 -1 -10.[illegible]	100 -1 -[illegible].1	20 -1 -6.3	40 -1 -[illegible].5	-1 -.9	...	...	0
(3)	2[illegible]-2.7θ -1 -9.8	...	-1 -[illegible].9	30 -1 -[illegible].2	-1 -.1	...	...	0
(4)	-1 -[illegible].2	...	-1 -2.1	-1 -2.1	...	...	...	0
(5)	-1 -2.6	...	-1 -.7	-1 -.7	...	...	...	0
Net	0	0	0	0	0	...	...	...
Im-plicit Prices v_j	9.8	[illegible]	4	3.6	1	0	...	...

Table 14 – Cycle 4

WORK SHEET FOR DETERMINING OPTIMAL ASSIGNMENT UNDER UNCERTAIN DEMAND

$\delta_{33} = -.9$, $\theta = 20/22 = .9$, Expected Cost = \$1,542,000

Type of Aircraft	Route (1) N. Y. to L. A. 1-stop	(2) N. Y. to L. A. 2-stop	(3) N. Y. to Dallas 0-stop	(4) N. Y. to Dallas 1-stop	(5) N. Y. to Boston 0-stop	(6) Surplus Aircraft	Aircraft Available	Implicit Prices u_i
(1) A	10 16 17	15 21	23 13	25 16	81 10	0 0	10	−139
(2) B	...	12.8 10 1[illegible]	.9 14 15	[illegible].3 14 14	47 9	0 0	19	− 40
(3) C	...	[illegible].3 [illegible] 10	...	7 9	20.7 29 6	0 0	2[illegible]	− 18
(4) D	8.3−θ 9 17	11 10	6.7+θ 22 17	17 15	55 10	0 0	15	− 7[illegible]
Increment (1)	200 −1 −13	[illegible]0 −1 −13	140 −1 −[illegible]	10 −1 −[illegible]	[illegible]80 −1 −1	...	***	0
(2)	20 −1 −10.[illegible]	100 −1 −[illegible]	20 −1 −[illegible].3	40 −1 −[illegible]	20 −1 −.9	...	***	
(3)	[illegible]−θ −1 [illegible]	...	+220 −1 [illegible]	30 −1 −[illegible].2	−1 −.1	...	***	0
(4)	−1 −[illegible]	...	−1 [illegible].1	−1 −2.1	...	...	***	0
(5)	−1 −2.[illegible]	...	−1 −.[illegible]	−1 −.7	...	...	***	0
Net	0	0	0	0	0	***	***	***
Implicit Prices v_j	9.8	5.5	[illegible]	3.[illegible]	.8	0	***	***

Table 15 -- Cycle 5
(Optimal)

WORK SHEET FOR DETERMINING OPTIMAL ASSIGNMENT UNDER UNCERTAIN DEMAND
Minimum Expected Cost $1,524,000

Type of Aircraft	Route (1) N. Y. to L. A. 1-stop	(2) N. Y. to L. A. 2-stop	(3) N. Y. to Dallas 0-stop	(4) N. Y. to Dallas 1-stop	(5) N. Y. to Boston 0-stop	(6) Surplus Aircraft	Aircraft Available	Implicit Prices u_i
(1) A	10 16 18	15 21	28 18	23 16	81 10	0 0	10	-139
(2) B	***	12.8 10 15	.9 14 16	5.3 15 14	57 9	0 0	19	- 40
(3) C	***	4.3 5 10	***	7 9	20.7 29 6	0 0	25	- 18
(4) D	7.4 9 17	11 16	7.6 22 17	17 15	55 10	0 0	15	- 71
Increment (1)	200 -1 -13	50 -1 -13	140 -1 -7	10 -1 -7	580 -1 -1	***	***	0
(2)	20 -1 -10.4	100 -1 -9.1	20 -1 -6.3	40 -1 -5.6	20 -1 -.9	***	***	0
(3)	7 -1 -9.8	***	20 -1 -4.9	30 -1 -4.2	-1 -.1	***	***	0
(4)	-1 -5.2	***	-1 -2.1	-1 -2.1	***	***	***	0
(5)	-1 -2.6	***	-1 -.7	-1 -.7	***	***	***	0
Net	0	0	0	0	0		***	***
Implicit Prices v_j	9.8	5.5	4	3.6	.8	0	***	***

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